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Target Space Pseudoduality Between Dual Symmetric Spaces*

Orlando Alvarez†

Department of Physics
University of Miami
P.O. Box 248046
Coral Gables, FL 33124 USA

Abstract

A set of on shell duality equations is proposed that leads to a map between strings moving on symmetric spaces with opposite curvatures. The transformation maps "waves" on a riemannian symmetric space M to "waves" on its dual riemannian symmetric space $\widetilde{M}$. This transformation preserves the energy momentum tensor though it is not a canonical transformation. The preservation of the energy momentum tensor has a natural geometrical interpretation. The transformation maps "particle-like solutions" into static "soliton-like solutions". The results presented here generalize earlier results of E. Ivanov.

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†email: oalvarez@miami.edu

1 Introduction

We take spacetime Σ to be two dimensional Minkowski space. Let $\sigma^{\pm} = \tau \pm \sigma$ be the standard lightcone coordinates and let φ be a massless free scalar field satisfying the wave equation $\partial^2_{+-}\varphi = 0$. The standard abelian duality transformations (for a review see [1]) are

$$\partial_+\tilde{\varphi} = +\partial_+\varphi \,, \tag{1.1}$$

$$\partial_-\tilde{\varphi} = -\partial_-\varphi \,. \tag{1.2}$$

The integrability conditions for the above are precisely the equations of motion $\partial^2_{+-}\varphi = 0$. This means that we can construct $\tilde{\varphi}(\sigma^+, \sigma^-)$ and verify that as a consequence of the above $\partial^2_{+-}\tilde{\varphi} = 0$. A generalization of the above are the pseudochiral models of the type introduced by Zakharov and Mikhailov [2]. Consider a standard sigma model with target space a Lie group G and with equations of motions $\partial^a(g^{-1}\partial_a g) = 0$. Introduce the dual Lie algebra valued field ϕ by

$$g^{-1}\partial_a g = -\epsilon_a{}^b \partial_b \phi \,.$$

It was shown by Nappi that these two descriptions were not quantum mechanically equivalent [3]. The correct dual models were found by Fridling and Jevicki [4] and by Fradkin and Tseytlin [5]. It was eventually understood that the duality transformation should be a canonical transformation [6, 7] and the transformation in the pseudochiral model is not. Nevertheless the pseudochiral model has a variety of interesting field theoretic features [6, 8]. Motivated by string theory, there is now a vast literature on nonabelian duality [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and Poisson-Lie duality [22, 23, 24, 25, 26, 27].

Following up on recent work [28] we consider a variant of the duality equations proposed there. These equations are a generalization of the ones of Zakharov and Mikhailov. In light of the introductory paragraph it will be interesting to study the physical and mathematical properties of these equations. Here we point out that there is an interesting transformation that maps solutions of the wave equation on a symmetric space into solutions of the wave equation on a symmetric space with the opposite curvature (see the next paragraph). We note that it was pointed out in [8] that the opposite signs found by Nappi in the beta functions for the dual models of Zakharov and Mikhailov are explained by observing that the generalized curvatures in the models have opposite signs.

The results presented here are a generalization[1] of results of E. Ivanov [29]. There

[1] I am thankful to E. Ivanov for bringing his work to my attention.

is a large body of literature discussing conserved currents in sigma models based on groups or coset spaces. A seminal work was Pohlmeyer's construction [30] for an infinite number of conserved currents in sigma models with target space S^n. This was generalized by Eichenherr and Forger [31, 32] who showed that the construction generalized to symmetric spaces. Ivanov expanded on these ideas and in doing so introduced the notion the *dual algebra* and the *dual sigma model*. He states,

> "… in which we show that the equations of any $d = 2$ σ model on a symmetric space simultaneously describe a $d = 2$ σ model on some other, dual factor space…" [29, p. 475].

He explicitly works out the example of a sigma model with target a compact real Lie group G (with zero B-field). He views G as a symmetric space $G \times G/G$ and he explicitly shows that the sigma model on G is dual to the sigma model on $G^{\mathbb{C}}/G$ where $G^{\mathbb{C}}$ is the complexification of G. He subsequently asserts that the construction generalizes to a generic symmetric space. Ivanov's construction for a symmetric space is given in Section 6. The work here generalizes Ivanov's in that we assume a general riemannian manifold and show that it must be a symmetric space.

We first establish some notation. The sigma model with target space M, metric g and 2-form B will be denoted by (M, g, B) and has lagrangian

$$\mathcal{L} = \frac{1}{2} g_{ij}(x) \left(\frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} - \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \sigma} \right) + B_{ij}(x) \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \sigma} \tag{1.3}$$

with canonical momentum density

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = g_{ij} \dot{x}^j + B_{ij} x'^j \ . \tag{1.4}$$

The stress energy tensor for this sigma model is given by $\Theta_{+-} = 0$,

$$\Theta_{++} = g_{ij}(x) \partial_+ x^i \partial_+ x^j \quad \text{and} \quad \Theta_{--} = g_{ij}(x) \partial_- x^i \partial_- x^j \ . \tag{1.5}$$

In general a duality transformation between sigma models (M, g, B) and $(\widetilde{M}, \tilde{g}, \widetilde{B})$ is a canonical transformation between the respective phase spaces that preserves the respective hamiltonian densities. We can study a less restrictive situation where we only have "on shell duality". By this we mean that we only require that a map exists between solutions to the equations of motion of (M, g, B) and solutions of the equations of motion of $(\widetilde{M}, \tilde{g}, \widetilde{B})$. The "on shell" transformation proposed below will be referred to as *pseudoduality*. There is a certain naturalness to the pseudoduality transformation because it preserves the stress energy tensor.

As in [28] it is convenient to choose an orthonormal frame $\{\omega^i\}$ with the antisymmetric riemannian connection ω_{ij}. The Cartan structural equations are

$$
\begin{aligned}
d\omega^i &= -\omega_{ij} \wedge \omega^j \, , \\
d\omega_{ij} &= -\omega_{ik} \wedge \omega_{kj} + \frac{1}{2} R_{ijkl} \omega^k \wedge \omega^l \, .
\end{aligned}
$$

We also define the curvature 2-forms by $\Omega_{ij} = \frac{1}{2} R_{ijkl}\omega^k \wedge \omega^l$. Consider two sigma models (M, g, B) and $(\widetilde{M}, \tilde{g}, \widetilde{B})$. Using the orthonormal frame we define the appropriate σ^a derivative of the maps $x : \Sigma \to M$ and $\tilde{x} : \Sigma \to \widetilde{M}$ by[2]

$$
\omega^i = x^i{}_a d\sigma^a \quad \text{and} \quad \tilde{\omega}^i = \tilde{x}^i{}_a d\sigma^a \, . \tag{1.6}
$$

In order to describe the equations of motion for the sigma model we need to consider second derivatives. The covariant derivatives of $x^i{}_a$ are $x^i{}_{ab}$ and are defined by

$$
dx^i{}_a + \omega_{ij} x^j{}_a = x^i{}_{ab} d\sigma^b \, . \tag{1.7}
$$

Taking the exterior derivative of $\omega^i = x^i{}_a d\sigma^a$ we learn that $x^i{}_{ab} = x^i{}_{ba}$. The equations of motion coming from (1.3) are

$$
x^k{}_{+-} = -\frac{1}{2} H_{kij} x^i{}_+ x^j{}_- \, , \tag{1.8}
$$

where $H = dB$. There are similar definitions and equations for $\tilde{x}^i$.

The lagrangian version of the hamiltonian duality equations in [28] may be written as

$$
\tilde{x}_+(\sigma) = +T_+(x, \tilde{x}) x_+(\sigma) \, , \tag{1.9}
$$

$$
\tilde{x}_-(\sigma) = -T_-(x, \tilde{x}) x_-(\sigma) \, . \tag{1.10}
$$

The orthogonal matrix valued functions $T_\pm : M \times \widetilde{M} \to \mathrm{SO}(n)$ are not arbitrary but related by

$$
T_+(I + n) = T_-(I - n) \, , \tag{1.11}
$$

where the antisymmetric tensor n_{ij} on $M \times \widetilde{M}$ satisfies some PDEs given in [28]. In this article we relax such restrictions on $T_\pm$ and consider orthogonal matrix valued functions $T_\pm : \Sigma \to \mathrm{SO}(n)$ with the constraint that solutions to the the sigma model (M, g, B) are mapped into solutions of $(\widetilde{M}, \tilde{g}, \widetilde{B})$ and vice versa. We will refer to these equations as the *pseudoduality equations*. Note that the pseudoduality transformations

[2] More correctly these equations are pullbacks of the type $x^*\omega^i = x^i{}_a d\sigma^a$. In the field of exterior differential systems [33] the pullback is implicit and not usually written. We adhere to that convention.

satisfy $\widetilde{\Theta}_{\pm\pm} = \Theta_{\pm\pm}$. This is very different than in [28] where $T_\pm$ are functions on $M \times \widetilde{M}$. In other words, we allow explicit dependence on σ in T where as in [28] the dependence occurs because x and $\tilde{x}$ depend on σ.

We specialize in this article to the case where $T_+ = T_- = T : \Sigma \to \mathrm{SO}(n)$ and $H = 0$, $\widetilde{H} = 0$. In this case the equations of motion become[3] $x_{+-} = \tilde{x}_{+-} = 0$ and the duality equations are

$$\tilde{x}_\pm(\sigma) = \pm T(\sigma) x_\pm(\sigma)\,. \tag{1.12}$$

We point out that when $H = \widetilde{H} = 0$ the sigma models are worldsheet parity and worldsheet time reversal invariant. The isometry T is odd under worldsheet parity and under worldsheet time reversal. Note that in the pseudochiral model $\widetilde{H} \neq 0$ and therefore it is not covered in this paper. The general case with generic H and $\widetilde{H}$ will be treated elsewhere [34].

Duality equations (1.12) are mathematically quite interesting. We digress a bit and discuss the question of local riemannian isometries. Assume we have a map $f : M \to \widetilde{M}$ and we wish for this map to be an isometry. If $\{\omega^j\}$ and $\{\tilde{\omega}^i\}$ are local orthonormal frames then we require that the metric be preserved: $f^*(\tilde{\omega}^i \otimes \tilde{\omega}^i) = \omega^j \otimes \omega^j$. The solution to this equation is $f^*\tilde{\omega} = T\omega$ where $T : M \to \mathrm{O}(n)$. This Pfaffian system of equations is integrable [33] if the Riemann curvature tensor and its higher order covariant derivatives in the two spaces agree when identified via T. For more details see the discussion in the paragraph following (3.16). In our case we begin with maps $x : \Sigma \to M$, $\tilde{x} : \Sigma \to \widetilde{M}$ that satisfy hyperbolic Lorentz invariant equations. We fix a Lorentz frame and we observe that we are interested in maps from $\{x : \Sigma \to M \,|\, x_{+-} = 0\}$ into $\{\tilde{x} : \Sigma \to \widetilde{M} \,|\, \tilde{x}_{+-} = 0\}$ that preserve the two independent components of the energy momentum tensor Θ_{++} and Θ_{--}. A linearity assumption and (1.5) tells us that the maps have to be of form (1.9) and (1.10) with the more general $T_\pm(\sigma)$. Given a map $x : \Sigma \to M$ there are two preferred tangent vector fields $\partial/\partial\sigma^+$ and $\partial/\partial\sigma^-$. The conditions that require Θ to be preserved are very geometric: the lengths of $\partial/\partial\sigma^+$ and $\partial/\partial\sigma^-$ are preserved by the map. If $*_\Sigma$ denotes the Hodge duality operation on Σ, our equations may be written as $\tilde{\omega} = *_\Sigma(T\omega)$ where we interpret ω and $\tilde{\omega}$ as pull backs to Σ. It is the desire to have the $*_\Sigma$ operation that introduces a -1 in (1.10) even though in principle the -1 could be absorbed[4] into T_-. Here we show that there are interesting maps between "waves" on M and "waves" on $\widetilde{M}$ that preserve natural geometrical structures. This may be of interest to researchers who study two dimensional wave equations.

[3] We remind the reader that solutions to Laplace's equation are called harmonic functions yet solutions to the wave equation are simply called "waves".

[4] We may have to allow $T_\pm \in \mathrm{O}(n)$.

This paper is organized in the following way. In Section 2 we discuss pseudoduality between the 2-sphere S^2 and the 2-dimensional hyperbolic space H^2 very concretely. In particular we explicitly verify the necessity for the the orthogonal matrix valued function T. In Section 3 we become a bit more abstract but still concrete by working with explicit metrics and show that there is pseudoduality between S^n and H^n. In Section 4 we assume general metrics and show that the manifolds M and $\widetilde{M}$ must be symmetric spaces with the "opposite curvatures". In Section 5 we construct many examples by discussing the theory of dual symmetric spaces. Section 7 is a discussion of the results of this article.

2 A Pedagogic Example

It is worthwhile to be very concrete and to consider the pseudoduality between strings moving on a 2-sphere S^2 and those moving on a 2-hyperboloid H^2. This example illustrates the necessity for the matrix T. We use a different coordinate version of the constant curvature metric than in Section 3 to emphasize that everything is independent of the choice of coordinates. The respective constant curvature metrics on S^2 and H^2 in polar normal coordinates are

$$\begin{aligned} ds^2 &= dr^2 + \frac{\sin^2 \alpha r}{\alpha^2} d\theta^2 \ , \\ d\tilde{s}^2 &= d\tilde{r}^2 + \frac{\sinh^2 \tilde{\alpha}\tilde{r}}{\tilde{\alpha}^2} d\tilde{\theta}^2 \ . \end{aligned}$$

The equations of motion for the sigma models are:

$$\partial^2_{+-} r = (\sin \alpha r \cos \alpha r)/\alpha \ \partial_+ \theta \partial_- \theta \ , \tag{2.1}$$

$$(\sin \alpha r)/\alpha \ \partial^2_{+-} \theta = -\cos \alpha r (\partial_+ r \partial_- \theta + \partial_- r \partial_+ \theta) \ , \tag{2.2}$$

$$\partial^2_{+-} \tilde{r} = (\sinh \tilde{\alpha}\tilde{r} \cosh \tilde{\alpha}\tilde{r})/\tilde{\alpha} \ \partial_+ \tilde{\theta} \partial_- \tilde{\theta} \ , \tag{2.3}$$

$$(\sinh \tilde{\alpha}\tilde{r})/\tilde{\alpha} \ \partial^2_{+-} \tilde{\theta} = -\cosh \tilde{\alpha}\tilde{r} (\partial_+ \tilde{r} \partial_- \tilde{\theta} + \partial_- \tilde{r} \partial_+ \tilde{\theta}) \ . \tag{2.4}$$

This elementary example illustrates the importance of the matrix T. Assume $T = I$ then two of the duality equations in this coordinate system would be $\partial_+ \tilde{r} = \partial_+ r$ and $\partial_- \tilde{r} = -\partial_- r$. This would imply that $\partial^2_{+-} r = 0$ and $\partial^2_{+-} \tilde{r} = 0$ but these are not the equations of motion. We will see that with a $T \neq I$ we can construct a pseudoduality transformation.

First we need an orthonormal frame. On S^2 we have $\omega^r = dr$ and $\omega^\theta = \sin \alpha r / \alpha \ d\theta$. From this it follows that the connection and the curvature are respectively given by

$\omega_{r\theta} = -\cos(\alpha r)\ d\theta$ and $d\omega_{r\theta} = \alpha^2 \omega^r \wedge \omega^\theta$. On H^2 we have $\tilde{\omega}^{\tilde{r}} = d\tilde{r}$ and $\tilde{\omega}^{\tilde{\theta}} = \sinh \alpha\tilde{r}/\alpha\ d\tilde{\theta}$. From this we see that the connection and the curvature are respectively given by $\tilde{\omega}_{\tilde{r}\tilde{\theta}} = -\cosh(\tilde{\alpha}\tilde{r})\ d\tilde{\theta}$ and $d\tilde{\omega}_{\tilde{r}\tilde{\theta}} = -\tilde{\alpha}^2 \tilde{\omega}^{\tilde{r}} \wedge \tilde{\omega}^{\tilde{\theta}}$.

We need a 2×2 orthogonal matrix that we parametrize by a rotation angle ϕ. The duality equations are

$$\begin{pmatrix} \partial_\pm \tilde{r} \\ \sinh(\tilde{\alpha}\tilde{r})/\tilde{\alpha}\ \partial_\pm \tilde{\theta} \end{pmatrix} = \pm \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \partial_\pm r \\ \sin(\alpha r)/\alpha\ \partial_\pm \theta \end{pmatrix} . \tag{2.5}$$

The integrability conditions on the above lead to the equation

$$d\phi = -\cosh(\tilde{\alpha}\tilde{r})\ d\tilde{\theta} + \cos(\alpha r)\ d\theta\,. \tag{2.6}$$

This is integrable if $\alpha = \tilde{\alpha}$ where *integrable* means integrable modulo the equations of motion.

What happens to the point particle geodesics on S^2? One easily verifies that for constant a, the functions $r = a \cdot (\sigma^+ + \sigma^-) = a \cdot (2\tau)$ and $\theta = 0$ give a solution of equations (2.1) and (2.2). This corresponds to "particle geodesics" on S^2. We would like to understand what are the dual solutions to the particle geodesics. If we note that $\partial_\pm r = a$ and $\partial_\pm \theta = 0$ we see that the duality equations become

$$\partial_\pm \tilde{r} = \pm a \cos\phi \tag{2.7}$$

$$\sinh(\tilde{\alpha}\tilde{r})/\tilde{\alpha}\ \partial_\pm \tilde{\theta} = \mp a \sin\phi \tag{2.8}$$

An immediate consequence of the above is that $(\partial_+ + \partial_-)\tilde{r} = 0$ and $(\partial_+ + \partial_-)\tilde{\theta} = 0$. We conclude that $\tilde{r} = \tilde{r}(\sigma^+ - \sigma^-) = \tilde{r}(2\sigma)$ and $\tilde{\theta} = \tilde{\theta}(\sigma^+ - \sigma^-) = \tilde{\theta}(2\sigma)$. Thus we get static solutions on the hyperboloid. Note that (2.6) immediately tells us that $\phi = \phi(\sigma^+ - \sigma^-) = \phi(2\sigma)$. Since everything is a function of $2\sigma = \sigma^+ - \sigma^-$ the situation reduces to functions of a single variable. The transformed solutions will be static solutions (τ independent). Note that equations (2.7) and (2.8) lead to

$$(\partial_+ \tilde{r})^2 + \left(\frac{\sinh \tilde{\alpha}\tilde{r}}{\tilde{\alpha}}\ \partial_+ \tilde{\theta} \right)^2 = a^2\,. \tag{2.9}$$

This is the "conservation of energy" equation associated with the particle lagrangian

$$L = \frac{1}{2} \left[(\partial_+ \tilde{r})^2 + \left(\frac{\sinh \tilde{\alpha}\tilde{r}}{\tilde{\alpha}}\ \partial_+ \tilde{\theta} \right)^2 \right]$$

This is a standard problem in classical mechanics. The canonical momentum

$$\tilde{J} = \frac{\partial L}{\partial(\partial_+ \tilde{\theta})} = \left(\frac{\sinh \tilde{\alpha}\tilde{r}}{\tilde{\alpha}} \right)^2\ \partial_+ \tilde{\theta}$$

is a constant of the motion. Thus the energy integral (2.9) may be written as

$$(\partial_+\tilde{r})^2 + \frac{\tilde{\alpha}^2\tilde{J}^2}{\sinh^2\tilde{\alpha}\tilde{r}} = a^2 \tag{2.10}$$

and this problem is reducible to quadrature. This is interesting because we know that there exists crystallographic subgroups Γ of $\mathrm{SL}(2,\mathbb{R})$ such that H/Γ is a genus $g > 1$ compact Riemann surface. Such a surface has closed geodesics of minimal length and some of these are the static "soliton-like" solutions $(\tilde{r}(2\sigma), \tilde{\theta}(2\sigma))$ we are constructing above.

3 Constant Curvature Metric

Before presenting the general theory we study the special case of constant curvature spaces. Again we take $T_+ = T_- = T$. The two dimensional nonlinear sigma model on a space with constant positive curvature $k > 0$ is shown to be pseudodual to the nonlinear sigma model on a space with constant negative curvature $-k$. The pseudoduality equations are used to map solutions of one model into solutions of the other model.

Locally, a metric on a space with constant curvature is may be written in the Poincaré form

$$ds^2 = \frac{dx^i \otimes dx^i}{(1+kx^2)^2} \; . \tag{3.1}$$

An orthonormal frame in this metric is

$$\omega^i = \frac{dx^i}{1+kx^2} \; . \tag{3.2}$$

The connection one-forms (with respect to the orthonormal frame) are

$$\omega_{ij} = 2k\,\frac{x^i dx^j - x^j dx^i}{1+kx^2} \; . \tag{3.3}$$

The first Cartan structural equation is $d\omega^i = -\omega_{ij} \wedge \omega^j$. The second structural equation gives the the curvature two forms

$$\Omega_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj} = \frac{1}{2} R_{ijkl}\omega^k \wedge \omega^l = 4k\,\omega^i \wedge \omega^j \; . \tag{3.4}$$

Technically the curvature is $4k$.

The lagrangian for the sigma model is

$$\mathcal{L} = \frac{2\partial_+ x^i \partial_- x^i}{(1+kx^2)^2} \; . \tag{3.5}$$

The equations of motion associated with this lagrangian are

$$\partial^2_{+-} x^i = 2k\, \frac{x^j\partial_+ x^j \partial_- x^i + x^j \partial_- x^j \partial_+ x^i - x^i \partial_+ x^j \partial_- x^j}{1+kx^2}\ . \tag{3.6}$$

Assume we have two constant curvature spaces M and $\widetilde{M}$ with respective curvatures k and $\tilde{k}$. We would like to see if it is possible to have a duality transformation between them. Assume we have a solution $x(\sigma)$ of the sigma model on M. We attempt to construct a solution of the sigma model on $\widetilde{M}$ by requiring that

$$\begin{aligned} \frac{\partial_+\tilde{x}^i}{1+\tilde{k}\tilde{x}^2} &= T^i{}_j\, \frac{\partial_+ x^j}{1+kx^2}\ , & (3.7)\\ \frac{\partial_-\tilde{x}^i}{1+\tilde{k}\tilde{x}^2} &= -T^i{}_j\, \frac{\partial_- x^j}{1+kx^2}\ , & (3.8) \end{aligned}$$

where T is an orthogonal matrix, $\det T = 1$. Note that the stress energy tensors satisfy $\Theta_{\pm\pm} = \widetilde{\Theta}_{\pm\pm}$.

The first step towards the integrability conditions for the above system is to differentiate (3.7) with respect to σ^- and (3.8) with respect to σ^+:

$$\begin{aligned} \frac{\partial_-(\partial_+\tilde{x}^i)}{1+\tilde{k}\tilde{x}^2} - 2\tilde{k}\frac{\tilde{x}^j\partial_-\tilde{x}^j\partial_+\tilde{x}^i}{(1+\tilde{k}\tilde{x}^2)^2} &= (\partial_- T^i{}_j)\frac{\partial_+ x^j}{1+kx^2} \\ &+ T^i{}_j\,\frac{\partial^2_{+-}x^j}{1+kx^2} - 2kT^i{}_j\,\frac{x^k\partial_- x^k\partial_+ x^j}{(1+kx^2)^2}\ , & (3.9)\\ -\frac{\partial_+(\partial_-\tilde{x}^i)}{1+\tilde{k}\tilde{x}^2} + 2\tilde{k}\frac{\tilde{x}^j\partial_+\tilde{x}^j\partial_-\tilde{x}^i}{(1+\tilde{k}\tilde{x}^2)^2} &= (\partial_+ T^i{}_j)\frac{\partial_- x^j}{1+kx^2} \\ &+ T^i{}_j\,\frac{\partial^2_{+-}x^j}{1+kx^2} - 2kT^i{}_j\,\frac{x^k\partial_+ x^k\partial_- x^j}{(1+kx^2)^2}\ . & (3.10) \end{aligned}$$

By imposing the integrability conditions $\partial_+(\partial_-\tilde{x}^i) = \partial_-(\partial_+\tilde{x}^i)$ we can add the above two equations to eliminate $\partial^2_{+-}\tilde{x}$ and by imposing *only* the equations of motion for x^i and the duality relations we obtain

$$\begin{aligned} (\partial_- T^i{}_j)\frac{\partial_+ x^j}{1+kx^2} + (\partial_+ T^i{}_j)\frac{\partial_- x^j}{1+kx^2} &= 2\tilde{k}\,\frac{\tilde{x}^j\partial_-\tilde{x}^i - \tilde{x}^i\partial_-\tilde{x}^j}{1+\tilde{k}\tilde{x}^2}\,T^j{}_k\,\frac{\partial_+ x^k}{1+kx^2} \\ &+ 2\tilde{k}\,\frac{\tilde{x}^j\partial_+\tilde{x}^i - \tilde{x}^i\partial_+\tilde{x}^j}{1+\tilde{k}\tilde{x}^2}\,T^j{}_k\,\frac{\partial_- x^k}{1+kx^2} \\ &- 2k\,T^i{}_j\,\frac{x^k\partial_- x^j - x^j\partial_- x^k}{1+kx^2}\,\frac{\partial_+ x^k}{1+kx^2} \\ &- 2k\,T^i{}_j\,\frac{x^k\partial_+ x^j - x^j\partial_+ x^k}{1+kx^2}\,\frac{\partial_- x^k}{1+kx^2}\ . & (3.11) \end{aligned}$$

The particular combinations in the above follow from the general theory and the form of (3.3). Note that at any point σ we can make $\partial_\pm x^i(\sigma)$ arbitrary thus we need that

$$\partial_- {T^i}_k = 2\tilde{k}\,\frac{\tilde{x}^j\partial_-\tilde{x}^i - \tilde{x}^i\partial_-\tilde{x}^j}{1+\tilde{k}\tilde{x}^2}\,{T^j}_k - 2k\,{T^i}_j\,\frac{x^k\partial_- x^j - x^j\partial_- x^k}{1+kx^2}\,, \tag{3.12}$$

$$\partial_+ {T^i}_k = 2\tilde{k}\,\frac{\tilde{x}^j\partial_+\tilde{x}^i - \tilde{x}^i\partial_+\tilde{x}^j}{1+\tilde{k}\tilde{x}^2}\,{T^j}_k - 2k\,{T^i}_j\,\frac{x^k\partial_+ x^j - x^j\partial_+ x^k}{1+kx^2}\,. \tag{3.13}$$

We interpret the above as the pullback under the map $\sigma \to (x(\sigma), \tilde{x}(\sigma))$ of the 1-form

$$d{T^i}_j = -\tilde{\omega}_{ik}{T^k}_j + {T^i}_k\omega_{kj}\ . \tag{3.14}$$

We can immediately verify that if one inserts (3.12) into equation (3.9) we get the equations for motion for $\tilde{x}$, *i.e.*, equations (3.6) with all quantities with tildes. Thus far we have shown that if one starts with a solution of the sigma model defined by lagrangian (3.5) and we impose the generalized duality equations then we find a solution to the "tilde" sigma model provided that we can satisfy equations (3.13) and (3.12).

The integrability conditions for the dT equation are found by taking the exterior derivative leading to

$$0 = -\widetilde{\Omega}_{ik}{T^k}_j + {T^i}_k\Omega_{kj} \tag{3.15}$$

In our case the curvature is very simple and the above reduces to

$$-\,4\tilde{k}\,\tilde{\omega}^i\wedge\tilde{\omega}^k{T^k}_j + 4k\,{T^i}_k\omega^k\wedge\omega^j = 0\ . \tag{3.16}$$

First we discuss what equation (3.15) *is not* and afterwards we discuss what *it is*. This equation shows up in the study of local riemannian isometries [33]. If we have two riemannian manifolds M and $\widetilde{M}$ and a map $f : M \to \widetilde{M}$ then the conditions that f be a local isometry is that there exists an orthogonal transformation T such that $\tilde{\omega}^i = {T^i}_j\omega^j$. When one works out the integrability conditions for this system one finds that (3.14) must be satisfied along with its integrability condition which is (3.15). This tells us that $\widetilde{R}_{ijkl}{T^k}_m{T^l}_n = {T^i}_k{T^j}_l R_{klmn}$. There are further integrability conditions which tell us that the covariant derivatives of the curvatures also satisfy similar relations. This is the classical theorem of Christoffel on local isometries between riemannian manifolds as reformulated by E. Cartan. In our constant curvature example the requirement simply becomes $k = \tilde{k}$.

Our situation is very different. In effect our setup[5] is a map from Σ to $M \times \widetilde{M}$. This tells us that

$$\omega^i = \frac{(\partial_a x^i)d\sigma^a}{1+kx^2} \quad \text{and} \quad \tilde{\omega}^i = \frac{(\partial_a \tilde{x}^i)d\sigma^a}{1+\tilde{k}\tilde{x}^2}\ .$$

[5] We avoid discussing jet bundles.

Substituting these into (3.16), using duality relations (3.7) and (3.8), and noting that at any (σ^+, σ^-) we are free to arbitrarily specify $\partial_\pm x^i(\sigma)$ leads to $k = -\tilde{k}$. The change in sign is due to the negative sign in (3.8). Thus we discover that the duality equations can be implemented if the constant curvature manifolds have the opposite curvature.

4 General Theory

This problem is best analyzed in the bundle of orthogonal frames [34]. Because most physicists are not familiar with this approach we work things out on the base manifold $M \times \widetilde{M}$. First thing to do is to take the exterior derivative of (1.12)

$$d\tilde{x}_\pm = \pm(dT)x_\pm \pm T dx_\pm$$

and use the definitions of the covariant derivative (1.7) to obtain

$$-\tilde{\omega}\tilde{x}_\pm + \tilde{x}_{\pm a} d\sigma^a = \pm(dT)x_\pm \mp T\omega x_\pm \pm T x_{\pm a} d\sigma^a \,.$$

If we use the duality equations (1.12) we have

$$\mp\tilde{\omega} T x_\pm + \tilde{x}_{\pm a} d\sigma^a = \pm(dT)x_\pm \mp T\omega x_\pm \pm T x_{\pm a} d\sigma^a \,.$$

A little algebra shows that

$$\tilde{x}_{\pm a} d\sigma^a = \pm(dT - T\omega + \tilde{\omega} T)x_\pm \pm T x_{\pm a} d\sigma^a \,. \tag{4.1}$$

We wish to isolate the integrability conditions so wedge the above with $d\sigma^\pm$.

$$\tilde{x}_{\pm\mp} d\sigma^\mp \wedge d\sigma^\pm = \pm(dT - T\omega + \tilde{\omega} T)x_\pm \wedge d\sigma^\pm \pm T x_{\pm\mp} d\sigma^\mp \wedge d\sigma^\pm \,.$$

We have two equations

$$\begin{aligned}
\tilde{x}_{+-} d\sigma^- \wedge d\sigma^+ &= +(dT - T\omega + \tilde{\omega} T)x_+ \wedge d\sigma^+ + T x_{+-} d\sigma^- \wedge d\sigma^+ \,, \\
\tilde{x}_{-+} d\sigma^+ \wedge d\sigma^- &= -(dT - T\omega + \tilde{\omega} T)x_- \wedge d\sigma^- - T x_{-+} d\sigma^+ \wedge d\sigma^- \,.
\end{aligned}$$

In principle we wish that the integrability conditions $\tilde{x}_{+-} = \tilde{x}_{-+}$ are satisfied if the equations of motion $x_{+-} = 0$ hold. Subsequently we would like that this implies that $\tilde{x}_{+-} = 0$. We might as well substitute $x_{+-} = 0$ and $\tilde{x}_{+-} = 0$ into the equations above and find

$$0 = (dT - T\omega + \tilde{\omega} T)x_\pm \wedge d\sigma^\pm \,.$$

Since $x^i_\pm$ may be arbitrarily specified at any σ we have

$$0 = (dT - T\omega + \tilde{\omega} T) \wedge d\sigma^\pm \,.$$

Since $x : \Sigma \to M$ and $\tilde{x} : \Sigma \to \widetilde{M}$ are maps of a two dimensional worldsheet we conclude that on the worldsheet, the covariant derivative of T vanishes

$$dT - T\omega + \tilde{\omega}T = 0\,. \tag{4.2}$$

The reason is that the covariant differential of T is a 1-form. All our objects arise from maps with domain Σ so the covariant differential of T is a 1-form on Σ. You are pulling back both ω and $\tilde{\omega}$ to Σ. Note that a covariantly constant tensor is determined its value at one point on the worldsheet; the values elsewhere are determined by parallel transport. In order to construct such a T we need to verify the integrability conditions for the above. These lead to important constraints on the geometry of M and $\widetilde{M}$. Taking the exterior derivative and using the Cartan structural equations leads to

$$-T^i{}_k\Omega_{kj} + \widetilde{\Omega}_{ik}T^k{}_j = 0$$

with special case (3.15). Expanding the above gives

$$-\frac{1}{2}T^i{}_k R_{kjlm}\omega^l \wedge \omega^m + \widetilde{R}_{iklm}T^k{}_j\tilde{\omega}^l \wedge \tilde{\omega}^m = 0\,.$$

If we now substitute (1.6) and use the pseudoduality equations (1.12) we see that

$$T^i{}_k T^j{}_l R_{klmn} = -\widetilde{R}_{ijkl}T^k{}_m T^l{}_n\,. \tag{4.3}$$

Thus we conclude that the manifolds M and $\widetilde{M}$ "have the opposite curvature". Next we take the exterior derivative of the above to look for further conditions. If the covariant differential of R is defined by $DR_{ijkl} = R_{ijkl;m}\omega^m$ and similarly for $\widetilde{R}$ we find

$$T^i{}_k T^j{}_l R_{klmn;p}\omega^p = -\widetilde{R}_{ijkl;p}T^k{}_m T^l{}_n\tilde{\omega}^p\,.$$

If we now substitute (1.6) into the above and use the pseudoduality equations (1.12) we obtain two equations since $d\sigma^+$ and $d\sigma^-$ are independent:

$$\begin{aligned} T^i{}_k T^j{}_l R_{klmn;q} &= -\widetilde{R}_{ijkl;p}T^k{}_m T^l{}_n T^p{}_q\,, \\ T^i{}_k T^j{}_l R_{klmn;q} &= +\widetilde{R}_{ijkl;p}T^k{}_m T^l{}_n T^p{}_q\,. \end{aligned}$$

The solution of the above is immediate

$$R_{klmn;q} = 0 \quad \text{and} \quad \widetilde{R}_{ijkl;p} = 0\,. \tag{4.4}$$

The manifolds M and $\widetilde{M}$ must be *locally symmetric spaces* with the "opposite curvature".

5 Dual Symmetric Spaces

There is a large class of examples of pairs of symmetric spaces with opposite curvature. These pairs are called dual symmetric spaces. The simplest pair is the n-sphere S^n and the n-dimensional hyperbolic space H^n. More complicated pairs are given by the Grassmann manifolds[6] $\mathrm{SO}(p+q)/\,\mathrm{SO}(p) \times \mathrm{SO}(q)$ and $\mathrm{O}_0(p,q)/\,\mathrm{SO}(p) \times \mathrm{SO}(q)$.

The general theory of symmetric spaces is very extensive [35, 36]. For our purposes we take a lighter approach [37, Chapter 11] and a more restrictive view and consider what are called *normal symmetric spaces*. These are specified by a triplet of data $(G/H, \sigma, Q)$ where G is a real Lie group, H is a closed subgroup of G and σ is an involutive automorphism of G. Let $\mathfrak{g}$ be the Lie algebra of G and denote the action of the automorphism σ on $\mathfrak{g}$ by s. Let $\mathfrak{h}$ be the $+1$ eigenspace of s and let $\mathfrak{m}$ be the -1 eigenspace of s. Since $\frac{1}{2}(I \pm s)$ are projectors we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The following are also required in a normal symmetric space:

1. If F is the fixed point set of σ and F_0 is its identity component then $F_0 \subset H \subset F$. This is a technical requirement that for our purposes we take it to mean that the Lie algebra of H is $\mathfrak{h}$.

2. Q is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. An ordinary symmetric space only requires Q to be an $\mathrm{Ad}(H)$-invariant inner product[7] on $\mathfrak{m}$. Here $\mathrm{Ad}(G)$ is the adjoint action of the group G on its Lie algebra $\mathfrak{g}$.

3. Q is s-invariant. This is not required for an ordinary symmetric space.

It follows that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$. The s-invariance of Q tells us that the direct sum decomposition is an orthogonal decomposition. Note that $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal with respect to any s-invariant quadratic form such as the Killing form $\mathrm{Tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$.

We pick an origin for the symmetric space G/H and associate $\mathfrak{m}$ with the tangent space at that point. For a normal symmetric space the sectional curvature associated to the 2-plane spanned by $X, Y \in \mathfrak{m}$ is given by the following simple formula [37, p. 319]

$$K(X,Y) = \frac{Q([X,Y],[X,Y])}{Q(X,X)Q(Y,Y) - Q(X,Y)^2} \tag{5.1}$$

that requires the $\mathrm{Ad}(G)$-invariance of Q in its derivation. We remind the reader that knowing all sectional curvatures is equivalent to knowing the curvature tensor [37] and

[6] $\mathrm{O}_0(p,q)$ is the component of $\mathrm{O}(p,q)$ connected to the identity.

[7] An $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$ leads to a G-invariant metric on M, see [37, p. 312].

that they are related by

$$K(X,Y) = \frac{R_{ijkl}X^iY^jX^kY^l}{Q(X,X)Q(Y,Y) - Q(X,Y)^2} \,.$$

Normal symmetric spaces $(G/H, \sigma, Q)$ and $(\widetilde{G}/\widetilde{H}, \tilde{\sigma}, \widetilde{Q})$ are said to be *dual symmetric spaces* if there exist

1. a Lie algebra isomorphism $S : \mathfrak{h} \to \tilde{\mathfrak{h}}$ such that $\widetilde{Q}(SV, SW) = -Q(V,W)$ for all $V, W \in \mathfrak{h}$.

2. a linear isometry $T : \mathfrak{m} \to \widetilde{\mathfrak{m}}$ such that $[TX, TY] = -S[X,Y]$ for all $X, Y \in \mathfrak{m}$.

In item (1) above the Lie algebra isomorphism tells us that brackets in $\mathfrak{h}$ are the same as in $\tilde{\mathfrak{h}}$. While the isometry in item (2) tells us that inner product on $\mathfrak{m}$ is the same as in $\widetilde{\mathfrak{m}}$.

For dual symmetric spaces it is easy to see that the sectional curvatures are related by $\widetilde{K}(TX, TY) = -K(X,Y)$.

It is worthwhile to work this out explicitly for the example of the dual symmetric spaces $\mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ and $\mathrm{O}_0(p,q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$. For $g \in G = \mathrm{SO}(p+q)$ or $g \in \widetilde{G} = \mathrm{O}_0(p,q)$ we take σ and $\tilde{\sigma}$ to be given by

$$g \mapsto \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} g \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} .$$

One easily verifies that $\mathfrak{h} = \tilde{\mathfrak{h}} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$ where the matrices are of the form

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

where $a \in \mathfrak{so}(p)$ and $c \in \mathfrak{so}(q)$. One also sees that $X \in \mathfrak{m}$ and $\widetilde{X} \in \widetilde{\mathfrak{m}}$ are of the form

$$X = \begin{pmatrix} 0 & -b^t \\ b & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{X} = \begin{pmatrix} 0 & b^t \\ b & 0 \end{pmatrix},$$

where b is an arbitrary $q \times p$ matrix. For the inner products we take $Q(X,Y) = -\frac{1}{2}\operatorname{Tr}(XY)$ and $\widetilde{Q}(\widetilde{X}, \widetilde{Y}) = +\frac{1}{2}\operatorname{Tr}(\widetilde{X}\widetilde{Y})$. These will be "riemannian" dual symmetric spaces because the metrics on $\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$ are positive definite. We take the subgroups H and $\widetilde{H}$ to be $\mathrm{SO}(p) \times \mathrm{SO}(q)$ and the Lie algebra isomorphism $S : \mathfrak{h} \to \tilde{\mathfrak{h}}$ is the identity transformation. The isometry $T : \mathfrak{m} \to \widetilde{\mathfrak{m}}$ is taken to be

$$T : \begin{pmatrix} 0 & -b^t \\ b & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & b^t \\ b & 0 \end{pmatrix} .$$

A brief computation shows that $[TX, TY] = -[X, Y]$. From this we see that $\mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ is a space with positive sectional curvature and $\mathrm{O}_0(p,q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ is a space with negative sectional curvature.

A more extensive discussion of dual symmetric spaces requires a thorough discussion of the theory of orthogonal involutive Lie algebras (orthogonal symmetric Lie algebras), see [35, 36].

6 Ivanov's Construction

In this article we have limited ourselves to studying sigma models on an arbitrary riemannian manifold with vanishing 3-form H_{ijk}. Ivanov analyzed a subset of those models. He studied sigma models that are Lie group theoretic in origin. In this case we can use special properties of Lie groups to simplify the analysis. Assume we have a normal symmetric space G/H as in Section 5. We can choose an orthonormal basis that respects the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. If $\{T_i\}$ is such a basis with Lie bracket relations

$$[T_i, T_j] = f^k{}_{ij} T_k$$

then the $\mathrm{Ad}(G)$-invariance of Q tells us that f_{ijk} are totally antisymmetric. The indices $a, b, c, \ldots$ are associated to the basis elements that are in $\mathfrak{h}$ and the indices $\alpha, \beta, \gamma, \ldots$ are associated to basis elements in $\mathfrak{m}$. With this notation the Lie algebra bracket relations are

$$[T_a, T_b] = f^c{}_{ab} T_c \,, \tag{6.1}$$

$$[T_a, T_\beta] = f^\gamma{}_{a\beta} T_\gamma \,, \tag{6.2}$$

$$[T_\alpha, T_\beta] = f^c{}_{\alpha\beta} T_c \,. \tag{6.3}$$

Let θ^i be the the Maurer-Cartan forms for the Lie group G that satisfy the Maurer-Cartan equations

$$d\theta^i + \frac{1}{2} f^i{}_{jk} \theta^j \wedge \theta^k = 0 \,.$$

Using the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ we may write the equations above as

$$d\theta^a + \frac{1}{2} f^a{}_{bc} \theta^b \wedge \theta^c = -\frac{1}{2} f^a{}_{\beta\gamma} \theta^\beta \wedge \theta^\gamma \,, \tag{6.4}$$

$$d\theta^\beta + f^\beta{}_{b\gamma} \theta^b \wedge \theta^\gamma = 0 \,. \tag{6.5}$$

It is worthwhile discussing the geometric meaning of the above equations. It is well known that G is a principal H-bundle over G/H. The H-connection $\theta^a T_a$ on G defines

the horizontal tangent spaces. Its curvature is given by the right hand side of (6.4). Equation (6.5) states that the covariant differential of the $\mathfrak{m}$-valued 1-form $\theta^\beta T_\beta$ is zero.

The equations of motion nonlinear sigma model with target space G/H may be formulated in terms of a map $g : \Sigma \to G$ that satisfies the equations

$$\begin{aligned} d\theta^a + \frac{1}{2} f^a{}_{bc}\theta^b \wedge \theta^c &= -\frac{1}{2} f^a{}_{\beta\gamma}\theta^\beta \wedge \theta^\gamma \,, && (6.6) \\ d\theta^\beta + f^\beta{}_{b\gamma}\theta^b \wedge \theta^\gamma &= 0 \,, && (6.7) \\ d(*_\Sigma \theta^\beta) + f^\beta{}_{b\gamma}\theta^b \wedge (*_\Sigma \theta^\gamma) &= 0 \,. && (6.8) \end{aligned}$$

Here we implicitly interpret θ^i as the pullback under $g : \Sigma \to G/H$. More properly we should have written $g^*\theta^i$. We use $*_\Sigma$ to denote the Hodge duality operation on Σ. On 1-forms it is given by $*_\Sigma(d\sigma^\pm) = \pm d\sigma^\pm$. The first two equations above are just the pullbacks to Σ of the Maurer-Cartan equations for G. The third equation (6.8) is essentially the wave equation. For future reference we note that if α, β are 1-forms on Σ then

$$(*_\Sigma \alpha) \wedge (*_\Sigma \beta) = -\alpha \wedge \beta \,, \qquad (6.9)$$

and that $(*_\Sigma)^2 \alpha = \alpha$.

Ivanov observed that if we define $\tilde{\theta}^\beta = *_\Sigma \theta^\beta$ and $\tilde{\theta}^a = \theta^a$ then the equations above become

$$\begin{aligned} d\tilde{\theta}^a + \frac{1}{2} f^a{}_{bc}\tilde{\theta}^b \wedge \tilde{\theta}^c &= +\frac{1}{2} f^a{}_{\beta\gamma}\tilde{\theta}^\beta \wedge \tilde{\theta}^\gamma \,, && (6.10) \\ d\tilde{\theta}^\beta + f^\beta{}_{b\gamma}\tilde{\theta}^b \wedge \tilde{\theta}^\gamma &= 0 \,, && (6.11) \\ d(*_\Sigma \tilde{\theta}^\beta) + f^\beta{}_{b\gamma}\tilde{\theta}^b \wedge (*_\Sigma \tilde{\theta}^\gamma) &= 0 \,. && (6.12) \end{aligned}$$

These equations may be interpreted as the equations of motion for a sigma model on the symmetric space $\widetilde{G}/H$ where $\widetilde{G}$ is a real Lie group with real Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{h} \oplus (i\mathfrak{m})$. This Lie algebra has Lie brackets given by

$$\begin{aligned} [\widetilde{T}_a, \widetilde{T}_b] &= f^c{}_{ab}\widetilde{T}_c \,, && (6.13) \\ [\widetilde{T}_a, \widetilde{T}_\beta] &= f^\gamma{}_{a\beta}\widetilde{T}_\gamma \,, && (6.14) \\ [\widetilde{T}_\alpha, \widetilde{T}_\beta] &= -f^c{}_{\alpha\beta}\widetilde{T}_c \,. && (6.15) \end{aligned}$$

In the above we have $\widetilde{T}_a = T_a$ and $\widetilde{T}_\beta = iT_\beta$. Note that if you think of $\widetilde{G}$ as a principal H-bundle then the curvature of $\widetilde{G}$, given by (6.10), is "opposite" to that of G, given by (6.6).

7 Discussion

In [28] it was shown that a necessary condition for target space duality is that the target spaces be parallelizable manifolds. In the scenario presented here where we only require "on shell" pseudoduality and we see that the parallelizable requirement is weakened substantially but we still find natural geometric restrictions on the target spaces. In the models considered here where the 3-forms H and $\widetilde{H}$ vanish we saw that pseudoduality requires that the target spaces be symmetric spaces with the "opposite curvature" generalizing a result of Ivanov [29]. The class of riemannian dual symmetric spaces provides a wealth of examples. In particular we studied explicitly the example of dual Grassmann manifolds $\mathrm{SO}(p+q)/\,\mathrm{SO}(p)\times\mathrm{SO}(q)$ and $\mathrm{O}_0(p+q)/\,\mathrm{SO}(p)\times\mathrm{SO}(q)$. We also saw the importance of the isometry T. Note that we never explicitly solved for T but we discovered conditions that guarantee its existence. Finally it should be emphasized that T depends on the path since it comes from integrating (4.2) along the path.

We note that equations (1.12) may also be written as

$$\tilde{x}_\tau = T x_\sigma\,, \tag{7.1}$$

$$\tilde{x}_\sigma = T x_\tau\,. \tag{7.2}$$

Thus we immediately see that "particle-like" solutions (σ independent) on M get mapped into static "soliton-like" solutions on $\widetilde{M}$ and vice-versa. If say $\widetilde{M}$ has noncontractible loops then there will be stable "soliton-like" solutions.

Recently Evans and Mountain [38] constructed an infinite number of local conserved commuting charges on sigma models with the target space being a compact symmetric space. It would be interesting to apply the results of this paper to the construction of Evans and Mountain.

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